Bits in two different time slots are \textit{statistically independent}.

\textit{a priori} probabilities: \( P[b_k = 0] = P_1, \ P[b_k = 1] = P_2 \).

Signals \( s_1(t) \) and \( s_2(t) \) have a duration of \( T_b \) seconds and finite energies:

\[
E_1 = \int_0^{T_b} s_1^2(t) \, dt, \quad E_2 = \int_0^{T_b} s_2^2(t) \, dt.
\]

Noise \( w(t) \) is stationary \textit{Gaussian}, zero-mean \textit{white} noise with two-sided power spectral density of \( N_0/2 \) (watts/Hz):

\[
E\{w(t)\} = 0, \ E\{w(t)w(t + \tau)\} = \frac{N_0}{2} \delta(\tau), \ w(t) \sim \mathcal{N}\left(0, \frac{N_0}{2}\right).
\]
Received signal over \([(k-1)T_b, kT_b]\):

\[
    r(t) = s_i(t - (k-1)T_b) + w(t), \quad (k-1)T_b \leq t \leq kT_b.
\]

Objective is to design a receiver (or demodulator) such that the probability of making an error is minimized.

Shall reduce the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).
Can you identify the signal sets \( \{ s_1(t), s_2(t) \} \)?

- **NRZ** (Transmit and Receive)
- **OOK** (Transmit and Receive)
Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as *linear combinations* of two *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$.

$\phi_1(t)$ and $\phi_2(t)$ form a set of *orthonormal* basis functions if and only if:

$\phi_1(t)$ and $\phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)dt = 0 \text{ (orthogonality),}$$

$$\int_0^{T_b} \phi_1^2(t)dt = \int_0^{T_b} \phi_2^2(t)dt = 1 \text{ (normalized to have unit energy).}$$

If $\{\phi_1(t), \phi_2(t)\}$ can be found, the representations are

$$s_1(t) = s_{11} \phi_1(t) + s_{12} \phi_2(t),$$

$$s_2(t) = s_{21} \phi_1(t) + s_{22} \phi_2(t).$$

It can be checked that the coefficients $s_{ij}$ can be calculated as follows:

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)dt, \quad i, j \in \{1, 2\},$$

An important question is: Given the signal set $s_1(t)$ and $s_2(t)$, can one always find an orthonormal basis functions to represent $\{s_1(t), s_2(t)\}$ exactly? If the answer is YES, is the set of orthonormal basis functions *UNIQUE*?
Geometric Representation of Signals $s_1(t)$ and $s_2(t)$ (II)

Provided that $\phi_1(t)$ and $\phi_2(t)$ can be found, the signals (which are waveforms) can be represented as vectors in a vector space (or signal space) spanned (i.e., defined) by the orthonormal basis set $\{\phi_1(t), \phi_2(t)\}$.

\[
\begin{align*}
  s_1(t) &= s_{11} \phi_1(t) + s_{12} \phi_2(t), \\
  s_2(t) &= s_{21} \phi_1(t) + s_{22} \phi_2(t), \\
  s_{ij} &= \int_0^{T_b} s_i(t) \phi_j(t) \, dt, \quad i, j \in \{1, 2\}, \\
  E_i &= \int_0^{T_b} s_i^2(t) \, dt = s_{i1}^2 + s_{i2}^2, \quad i \in \{1, 2\}, \\
  d_{12} = d_{21} &= \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 \, dt}
\end{align*}
\]

- $\int_0^{T_b} s_i(t) \phi_j(t) \, dt$ is the projection of signal $s_i(t)$ onto basis function $\phi_j(t)$.
- The length of a signal vector equals to the square root of its energy.
- It is always possible to find orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ to represent $s_1(t)$ and $s_2(t)$ exactly. In fact, there are infinite number of choices!
Gram-Schmidt Procedure

Gram-Schmidt (G-S) procedure is one method to find a set of orthonormal basis functions for a given arbitrary set of waveforms.

1. Let \( \phi_1(t) \equiv \frac{s_1(t)}{\sqrt{E_1}} \). Note that \( s_{11} = \sqrt{E_1} \) and \( s_{12} = 0 \).

2. Project \( s'_2(t) = \frac{s_2(t)}{\sqrt{E_2}} \) onto \( \phi_1(t) \) to obtain the correlation coefficient:

\[
\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}} \phi_1(t) dt = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) dt.
\]

3. Subtract \( \rho \phi_1(t) \) from \( s'_2(t) \) to obtain \( \phi'_2(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho \phi_1(t) \).

4. Finally, normalize \( \phi'_2(t) \) to obtain:

\[
\phi_2(t) = \frac{\phi'_2(t)}{\sqrt{\int_0^{T_b} \left[ \phi'_2(t) \right]^2 dt}} = \frac{\phi'_2(t)}{\sqrt{1 - \rho^2}}
\]

\[
= \frac{1}{\sqrt{1 - \rho^2}} \left[ \frac{s_2(t)}{\sqrt{E_2}} - \rho \frac{s_1(t)}{\sqrt{E_1}} \right].
\]
Gram-Schmidt Procedure: Summary

\[ \phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}, \]
\[ \phi_2(t) = \frac{1}{\sqrt{1 - \rho^2}} \left[ \frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}} \right], \]
\[ s_{21} = \int_0^{T_b} s_2(t) \phi_1(t) dt = \rho \sqrt{E_2}, \]
\[ s_{22} = \left( \sqrt{1 - \rho^2} \right) \sqrt{E_2}, \]
\[ d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 dt} = E_1 - 2\rho \sqrt{E_1 E_2 + E_2}. \]
Example 1

(a) Signal set. (b) Orthonormal function. (c) Signal space representation.
Example 2

\[ s_1(t) \]

\[ s_2(t) \]

\[ \phi_1(t) \]

\[ \phi_2(t) \]
Example 3

\[ s_1(t) = \begin{cases} V & 0 \leq t < T_b \\ -V & -\sqrt{E} \leq t < 0 \end{cases} \]

\[ s_2(t) = \begin{cases} V & 0 \leq t < T_b \\ -V & 0 \leq t < -\sqrt{E} \end{cases} \]

\[ \phi_2(t, \alpha) = \begin{cases} \frac{T_b}{2} & \alpha = \frac{T_b}{2} \\ \sqrt{E} & \alpha = 0 \end{cases} \]

\[ \rho = \frac{1}{E} \int_0^{T_b} s_2(t) s_1(t) \, dt \]

\[ = \frac{1}{V^2 T_b} \left[ V^2 \alpha - V^2 (T_b - \alpha) \right] \]

\[ = \frac{2\alpha}{T_b} - 1 \]

\[ \alpha = 0, \quad \frac{T_b}{4} \]
Example 4

(a) \(s_1(t)\)

\[
\begin{align*}
0 & \quad T_b \\
\text{V} & \quad \text{V}
\end{align*}
\]

(b) \(s_2(t)\)

\[
\begin{align*}
0 & \quad T_b/2 \quad T_b \\
\sqrt{3V} & \quad \sqrt{3V}
\end{align*}
\]

(a) \(\phi_1(t)\)

\[
\begin{align*}
0 & \quad T_b \\
1/\sqrt{T_b} & \quad 1/\sqrt{T_b}
\end{align*}
\]

(b) \(\phi_2(t)\)

\[
\begin{align*}
0 & \quad T_b/2 \quad T_b \\
\sqrt{3/\sqrt{T_b}} & \quad -\sqrt{3/\sqrt{T_b}}
\end{align*}
\]
\[ \rho = \frac{1}{E} \int_0^{T_b} s_2(t)s_1(t)dt = \frac{2}{E} \int_0^{T_b/2} \left( \frac{2\sqrt{3}}{T_b} Vt \right) V dt = \frac{\sqrt{3}}{2}, \]

\[ \phi_2(t) = \frac{1}{(1 - \frac{3}{4})^{\frac{1}{2}}} \left[ \frac{s_2(t)}{\sqrt{E}} - \rho \frac{s_1(t)}{\sqrt{E}} \right] = \frac{2}{\sqrt{E}} \left[ s_2(t) - \frac{\sqrt{3}}{2} s_1(t) \right], \]

\[ s_{21} = \frac{\sqrt{3}}{2} \sqrt{E}, \quad s_{22} = \frac{1}{2} \sqrt{E}. \]

\[ d_{21} = \left[ \int_0^{T_b} [s_2(t) - s_1(t)]^2 dt \right]^{\frac{1}{2}} = \sqrt{\left(2 - \sqrt{3}\right) E}. \]
Example 5

\[ s_1(t) = \sqrt{E} \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), \]
\[ s_2(t) = \sqrt{E} \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t + \theta). \]

where \( f_c = \frac{k}{2T_b} \), \( k \) an integer.

locus of \( s_2(t) \) as \( \theta \) varies from 0 to \( 2\pi \).
Obtaining Different Basis Sets by Rotation

(a) Arbitrary rotation

\[
\begin{bmatrix}
\hat{\phi}_1(t) \\
\hat{\phi}_2(t)
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\phi_1(t) \\
\phi_2(t)
\end{bmatrix}.
\]

(b) Rotation to make \( \hat{s}_{11} = \hat{s}_{21} \)

(a) Show that, regardless of the angle \( \theta \), the set \( \{ \hat{\phi}_1(t), \hat{\phi}_2(t) \} \) is also an orthonormal basis set.

(b) What are the values of \( \theta \) that make \( \hat{\phi}_1(t) \) perpendicular to the line joining \( s_1(t) \) to \( s_2(t) \)? For these values of \( \theta \), mathematically show that the components of \( s_1(t) \) and \( s_2(t) \) along \( \hat{\phi}_1(t) \), namely \( \hat{s}_{11} \) and \( \hat{s}_{21} \), are identical.

Remark: Rotating counter-clockwise for positive \( \theta \) and clock-wise for negative \( \theta \).
Example: Rotating \( \{ \phi_1(t), \phi_2(t) \} \) by \( \theta = 60^\circ \) to Obtain \( \{ \hat{\phi}_1(t), \hat{\phi}_2(t) \} \)

\[
\begin{bmatrix}
\hat{\phi}_1(t) \\
\hat{\phi}_2(t)
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\phi_1(t) \\
\phi_2(t)
\end{bmatrix} \Rightarrow \begin{cases}
\hat{\phi}_1(t) = \cos \theta \times \phi_1(t) + \sin \theta \times \phi_2(t) \\
\hat{\phi}_2(t) = -\sin \theta \times \phi_1(t) + \cos \theta \times \phi_2(t)
\end{cases}
\]

\( \hat{\phi}_1(t) \)

\[
\begin{array}{c}
\phi_1(t) \\
\hline
0 & 1 & t
\end{array}
\]

\[
\begin{array}{c}
\phi_2(t) \\
\hline
0 & 0.5 & 1 & t
\end{array}
\]

\( \hat{\phi}_1(t) \)

\[
\begin{array}{c}
\frac{1 + \sqrt{3}}{2} \\
0 & 0.5 & 1 & t
\end{array}
\]

\[
\begin{array}{c}
\frac{1 - \sqrt{3}}{2} \\
0 & 0.5 & 1 & t
\end{array}
\]

\( \hat{\phi}_2(t) \)

\[
\begin{array}{c}
\frac{1 - \sqrt{3}}{2} \\
0 & 0.5 & 1 & t
\end{array}
\]

\[
\begin{array}{c}
\frac{1 + \sqrt{3}}{2} \\
0 & 0.5 & 1 & t
\end{array}
\]
Gram-Schmidt Procedure for $M$ Waveforms $\{s_i(t)\}_{i=1}^{M}$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t) dt}},$$

$$\phi_i(t) = \frac{\phi'_i(t)}{\sqrt{\int_{-\infty}^{\infty} [\phi'_i(t)]^2 dt}}, \quad i = 2, 3, \ldots, N,$$

$$\phi'_i(t) = \frac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1} \rho_{ij} \phi_j(t),$$

$$\rho_{ij} = \int_{-\infty}^{\infty} \frac{s_i(t)}{\sqrt{E_i}} \phi_j(t) dt, \quad j = 1, 2, \ldots, i - 1.$$ 

If the waveforms $\{s_i(t)\}_{i=1}^{M}$ form a linearly independent set, then $N = M$. Otherwise $N < M$. 
Example: Find a Basis Set for the Following $M = 4$ Waveforms
Answer Found by Applying the G-S Procedure

\[
\begin{bmatrix}
  s_1(t) \\
  s_2(t) \\
  s_3(t) \\
  s_4(t)
\end{bmatrix} = \begin{bmatrix}
  \sqrt{2} & 0 & 0 \\
  0 & \sqrt{2} & 0 \\
  \sqrt{2} & 0 & 1 \\
  \sqrt{2} & 0 & 1
\end{bmatrix} \begin{bmatrix}
  \phi_1(t) \\
  \phi_2(t) \\
  \phi_3(t)
\end{bmatrix}
\]
Representation of Noise with Walsh Functions

Exact representation of noise using 4 Walsh functions is not possible.
Exact representation might be possible by using many more Walsh functions.
Can also use sine and cosine functions (Fourier representation).
Chapter 5: Optimum Receiver for Binary Data Transmission

Representation of Noise

- To represent noise $w(t)$, need to use a complete set of orthonormal functions:

$$w(t) = \sum_{i=1}^{\infty} w_i \phi_i(t), \quad \text{where} \quad w_i = \int_0^{T_b} w(t) \phi_i(t) \, dt.$$

- The coefficients $w_i$'s are random variables and understanding their statistical properties is imperative in developing the optimum receiver.

- Of course, the statistical properties of random variables $w_i$'s depend on the statistical properties of the noise $w(t)$, which is a random process.

- In communications, a major source of noise is thermal noise, which is modelled as Additive White Gaussian Noise (AWGN):
  - **White**: The power spectral density (PSD) is a constant (i.e., flat) over all frequencies.
  - **Gaussian**: The probability density function (pdf) of the noise amplitude at any given time follows a Gaussian distribution.

- When $w(t)$ is modelled as AWGN, the projection of $w(t)$ on each basis function, $w_i = \int_0^{T_b} w(t) \phi_i(t) \, dt$, is a Gaussian random variable (this can be proved).

- For zero-mean and white noise $w(t)$, $w_1$, $w_2$, $w_3$, ... are zero-mean and uncorrelated random variables:
  - $E\{w_i\} = E\left\{ \int_0^{T_b} w(t) \phi_i(t) \, dt \right\} = \int_0^{T_b} E\{w(t)\} \phi_i(t) \, dt = 0$.
  - $E\{w_i w_j\} = E\left\{ \int_0^{T_b} d\lambda w(\lambda) \phi_i(\lambda) \int_0^{T_b} d\tau w(\tau) \phi_j(\tau) \right\} = \begin{cases} \frac{N_0}{2}, & i = j \\ 0, & i \neq j \end{cases}$.

- Since $w(t)$ is not only zero-mean and white, but also Gaussian $\Rightarrow \{w_1, w_2, \ldots\}$ are Gaussian and statistically independent!!!
Need to Review Probability Theory & Random Processes – Chapter 3 Slides
Chapter 5: Optimum Receiver for Binary Data Transmission

Observing a waveform ⇒ Observing a set of numbers

Choose $\phi_1(t)$ and $\phi_2(t)$ so that they can be used to represent the two signals $s_1(t)$ and $s_2(t)$ exactly. The remaining orthonormal basis functions are simply chosen to complete the set in order to represent noise exactly.

The decision can be based on the observations $r_1, r_2, r_3, r_4, \ldots$.

Note that $r_j$, for $j = 3, 4, 5, \ldots$, does not depend on which signal ($s_1(t)$ or $s_2(t)$) was transmitted.
The criterion is to minimize the bit error probability.

Consider only the first $n$ terms ($n$ can be very very large), $\vec{r} = \{r_1, r_2, \ldots, r_n\}$

$\Rightarrow$ Need to partition the $n$-dimensional observation space into two decision regions, $\mathcal{R}_1$ and $\mathcal{R}_2$.

Decide a "1" was transmitted if $\vec{r}$ falls in this region.

Decide a "0" was transmitted if $\vec{r}$ falls in this region.
\[ P[\text{error}] = P[\text{("0" decided and "1" transmitted) or ("1" decided and "0" transmitted)}]. \]

\[ = P[0_D, 1_T] + P[1_D, 0_T] \]

\[ = P[0_D|1_T]P[1_T] + P[1_D|0_T]P[0_T] \]

\[ = P[\vec{r} \in \mathbb{R}_1|1_T]P_2 + P[\vec{r} \in \mathbb{R}_2|0_T]P_1 \]

\[ = P_2 \int_{\mathbb{R}_1} f(\vec{r}|1_T) \, d\vec{r} + P_1 \int_{\mathbb{R}_2} f(\vec{r}|0_T) \, d\vec{r} \]

\[ = P_2 \int_{\mathbb{R} - \mathbb{R}_2} f(\vec{r}|1_T) \, d\vec{r} + P_1 \int_{\mathbb{R}_2} f(\vec{r}|0_T) \, d\vec{r} \]

\[ = P_2 \int_{\mathbb{R}} f(\vec{r}|1_T) \, d\vec{r} + \int_{\mathbb{R}_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)] \, d\vec{r} \]

\[ = P_2 + \int_{\mathbb{R}_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)] \, d\vec{r} \]

\[ = P_1 - \int_{\mathbb{R}_1} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)] \, d\vec{r}. \]

The minimum error probability decision rule is

\[
\begin{align*}
P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) &\geq 0 \quad \Rightarrow \quad \text{decide "0" (0}_D) \\
 P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) &< 0 \quad \Rightarrow \quad \text{decide "1" (1}_D).
\end{align*}
\]
Chapter 5: Optimum Receiver for Binary Data Transmission

Optimal decision rule: $g(\bar{r}) > 0 \Rightarrow 0_D$, $g(\bar{r}) < 0 \Rightarrow 1_D$

Pr[error] =

$P_2 + \int_{\mathfrak{R}_2} [P_1 f(\bar{r} | 1_T) - P_2 f(\bar{r} | 0_T)] d\bar{r}$

$= P_1 + \int_{\mathfrak{R}_1} [P_2 f(\bar{r} | 1_T) - P_2 f(\bar{r} | 0_T)] d\bar{r}$
Equivalently,

\[
\frac{f(\vec{r} | 1_T)}{f(\vec{r} | 0_T)} \begin{cases} 1_D & \text{if } \frac{P_1}{P_2} > 0_D \\ 0_D & \text{if } \frac{P_1}{P_2} < 1_D \end{cases}
\]

(1)

The expression \( \frac{f(\vec{r} | 1_T)}{f(\vec{r} | 0_T)} \) is called the *likelihood ratio*.

The decision rule in (1) was derived without specifying any statistical properties of the noise process \( w(t) \).

Simplified decision rule when the noise \( w(t) \) is zero-mean, white and Gaussian:

\[
(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \begin{cases} 1_D & \text{if } \frac{P_1}{P_2} > 0_D \\ 0_D & \text{if } \frac{P_1}{P_2} < 1_D \end{cases} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_0 \ln \left( \frac{P_1}{P_2} \right).
\]

For the special case of \( P_1 = P_2 \) (signals are equally likely):

\[
(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \begin{cases} 1_D & \text{if } \frac{P_1}{P_2} > 0_D \\ 0_D & \text{if } \frac{P_1}{P_2} < 1_D \end{cases} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.
\]

\( \Rightarrow \) *minimum-distance receiver!*
Chapter 5: Optimum Receiver for Binary Data Transmission

Minimum-Distance Receiver

\[
\min \left\{ \left( r_1 - s_{11} \right)^2 + \left( r_2 - s_{12} \right)^2 \right\} \quad \text{subject to} \quad d_1^2 \\
\min \left\{ \left( r_1 - s_{21} \right)^2 + \left( r_2 - s_{22} \right)^2 \right\} \quad \text{subject to} \quad d_2^2
\]

Choose \( s_2(t) \)

Choose \( s_1(t) \)
Chapter 5: Optimum Receiver for Binary Data Transmission

Correlation Receiver Implementation

1. \( \mathbf{r}(t) = s_1(t) + \mathbf{w}(t) \)
2. \( r_1(t) = \phi_1(t) \)
3. \( r_2(t) = \phi_2(t) \)
4. Compute
   \( (r_1 - s_1)^2 + (r_2 - s_2)^2 - N_0 \ln(P_i) \)
   for \( i = 1, 2 \)
   and choose the smallest

5. Form the dot product
   \( \mathbf{r} \cdot \mathbf{s}_i \)

6. Choose the largest
   \( \frac{N_0}{2} \ln(P_1) - \frac{E_1}{2} \)
   \( \frac{N_0}{2} \ln(P_2) - \frac{E_2}{2} \)
Chapter 5: Optimum Receiver for Binary Data Transmission

Receiver Implementation using Matched Filters

\[
\int_0^{T_2} (\phi(t)) \, dt = \int_0^{T_2} (\phi(t)) \, dt
\]

\[
\phi_1(t) = \int_0^{T_2} (\phi(t)) \, dt
\]

\[
\phi_2(t) = \int_0^{T_2} (\phi(t)) \, dt
\]

\[
h_1(t) = \phi_1(T_b - t)
\]

\[
h_2(t) = \phi_2(T_b - t)
\]
Example 5.6

(a) $s_1(t)$

(b) $s_2(t)$

(a) $\phi_1(t)$

(b) $\phi_2(t)$
\[ s_1(t) = \phi_1(t) + \frac{1}{2} \phi_2(t), \]
\[ s_2(t) = -\phi_1(t) + \phi_2(t). \]
For each value of the signal-to-noise ratio (SNR), Matlab simulation was conducted for transmitting/receiving 500 equally-likely bits.

\[ \frac{(E_1 + E_2)/2}{N_0} = 0.99 \text{ (dB)}; \ P[\text{error}]=0.1 \]

\[ \frac{(E_1 + E_2)/2}{N_0} = 6.17 \text{ (dB)}; \ P[\text{error}]=0.01 \]
Choose $s_2(t)$

Choose $s_1(t)$

Choose $s_2(t)$

Choose $s_1(t)$

Choose $s_2(t)$

Choose $s_1(t)$

(a) $P_1 = P_2 = 0.5$, (b) $P_1 = 0.25$, $P_2 = 0.75$. (c) $P_1 = 0.75$, $P_2 = 0.25.$
Example 5.7

\[ s_2(t) = \phi_1(t) + \phi_2(t), \]
\[ s_1(t) = \phi_1(t) - \phi_2(t). \]
\[ \frac{N_0}{4} \ln \left( \frac{P_1}{P_2} \right) \]

Choose \[ s_2(t) \]

Choose \[ s_1(t) \]
Chapter 5: Optimum Receiver for Binary Data Transmission

1. \( T = \frac{N_0}{4} \ln \left( \frac{P_1}{P_2} \right) \)

(a) \( r_2 \geq T \Rightarrow \) choose \( s_2(t) \)

(b) \( r_2 < T \Rightarrow \) choose \( s_1(t) \)
Always possible by choosing \( \hat{\phi}_1(t) \) and \( \hat{\phi}_2(t) \) such that one of the two basis functions is perpendicular to the line joining the two signals.

The optimum receiver is still the minimum-distance receiver. However the terms \( (\hat{r}_1 - \hat{s}_{11})^2 \) and \( (\hat{r}_1 - \hat{s}_{21})^2 \) are the same on both sides of the comparison and hence can be removed. This means that one does not need to compute \( \hat{r}_1 \):

\[
\begin{align*}
(\hat{r}_1 - \hat{s}_{11})^2 + (\hat{r}_2 - \hat{s}_{12})^2 &\quad \begin{cases} 1 \text{ if } D > 0 \\ 0 \text{ if } D < 0 \end{cases} \\
\left(\frac{\hat{s}_{22} + \hat{s}_{12}}{2}\right) + \left(\frac{N_0/2}{\hat{s}_{22} - \hat{s}_{12}}\right) \ln\left(\frac{P_1}{P_2}\right) &\equiv T.
\end{align*}
\]
\[
\hat{\phi}_2(t) = \frac{s_2(t) - s_1(t)}{(E_2 - 2\rho \sqrt{E_1 E_2} + E_1)^{\frac{1}{2}}}, \quad T \equiv \frac{s_{22} + s_{12}}{2} + \left(\frac{N_0}{2} \frac{N_0}{s_{22} - s_{12}}\right) \ln \left(\frac{P_1}{P_2}\right).
\]
Example 5.8

\[
\hat{\phi}_1(t) = \frac{1}{\sqrt{2}} [\phi_1(t) + \phi_2(t)],
\]

\[
\hat{\phi}_2(t) = \frac{1}{\sqrt{2}} [-\phi_1(t) + \phi_2(t)].
\]
Chapter 5: Optimum Receiver for Binary Data Transmission

\[ b^T t_0 \] 

\[ b^T 2 \]

\[ b^T 2^2 \]

\[ b^T 2^2 b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]

\[ b^T \]
Chapter 5: Optimum Receiver for Binary Data Transmission

Receiver Performance

To detect $b_k$, compare $\hat{r}_2 = \int_{(k-1)T_b}^{kT_b} r(t)\hat{\phi}_2(t)dt$ to the threshold

$$T = \frac{\hat{s}_{12} + \hat{s}_{22}}{2} + \frac{N_0}{2(\hat{s}_{22} - \hat{s}_{12})} \ln \left( \frac{P_1}{P_2} \right).$$

Decision boundary

$$P[\text{error}] = P[(0 \text{ transmitted and } 1 \text{ decided}) \text{ or } (1 \text{ transmitted and } 0 \text{ decided})] = P[(0_T, 1_D) \text{ or } (1_T, 0_D)].$$
\begin{align*}
P[\text{error}] &= P[0_T, 1_D] + P[1_T, 0_D] = P[1_D|0_T]P[0_T] + P[0_D|1_T]P[1_T] \\
&= P_1 \int_T^\infty f(\hat{r}_2|0_T) \, d\hat{r}_2 + P_2 \int_{-\infty}^T f(\hat{r}_2|1_T) \, d\hat{r}_2 \\
&= P_1 Q \left( \frac{T - \hat{s}_{12}}{\sqrt{N_0/2}} \right) + P_2 \left[ 1 - Q \left( \frac{T - \hat{s}_{22}}{\sqrt{N_0/2}} \right) \right].
\end{align*}
Chapter 5: Optimum Receiver for Binary Data Transmission

\[ Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp \left( -\frac{\lambda^2}{2} \right) \, d\lambda. \]
Performance when \( P_1 = P_2 \)

\[
P[\text{error}] = Q \left( \frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_0/2}} \right) = Q \left( \frac{\text{distance between the signals}}{2 \times \text{noise RMS value}} \right).
\]

- Probability of error decreases as either the two signals become more dissimilar (increasing the distances between them) or the noise power becomes less.
- To maximize the distance between the two signals one chooses them so that they are placed 180° from each other \( \Rightarrow s_2(t) = -s_1(t) \), i.e., antipodal signaling.
- The error probability does not depend on the signal shapes but only on the distance between them.
Example 5.9

(a) Determine and sketch the two signals $s_1(t)$ and $s_2(t)$. 
(b) The two signals $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits 0 and 1, respectively, over an additive white Gaussian noise (AWGN) channel. Clearly draw the decision boundary and the decision regions of the optimum receiver. Write the expression for the optimum decision rule.

(c) Find and sketch the two orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that the optimum receiver can be implemented using only the projection $\hat{r}_2$ of the received signal $r(t)$ onto the basis function $\hat{\phi}_2(t)$. Draw the block diagram of such a receiver that uses a matched filter.

(d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

\[
S_1^{(M)}(t) = s_1(t) - \left( \text{component of } s_1(t) \text{ along } \hat{\phi}_1(t) \right)
\]
\[
S_2^{(M)}(t) = s_2(t) - \left( \text{component of } s_2(t) \text{ along } \hat{\phi}_1(t) \right)
\]

Clearly identify the locations of $S_1^{(M)}(t)$ and $S_2^{(M)}(t)$ in the signal space diagram. What is the average energy of this signal set? Compare it to the average energy of the original set. Comment.
Chapter 5: Optimum Receiver for Binary Data Transmission

\[ s_1(t) \]

\[ s_2(t) \]
\[
\begin{bmatrix}
\hat{\phi}_1(t) \\
\hat{\phi}_2(t)
\end{bmatrix}
= \begin{bmatrix}
\cos(-\pi/4) & \sin(-\pi/4) \\
-\sin(-\pi/4) & \cos(-\pi/4)
\end{bmatrix}
\begin{bmatrix}
\phi_1(t) \\
\phi_2(t)
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
\phi_1(t) \\
\phi_2(t)
\end{bmatrix}.
\]

\[
\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) - \phi_2(t)], \quad \hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)].
\]
Antipodal Signalling

The pulse shaping filter $h_T = p(t)$ defines the power spectrum density of the transmitted signal, which can be shown to be proportional to $|P(f)|^2$.

The error performance, $P[\text{error}]$ only depends on the energy $E$ of $p(t)$ and noise PSD level $N_0$. Specifically, the distance between $s_1(t)$ and $s_2(t)$ is $2\sqrt{E}$ (you should show this for yourself, algebraically or geometrically). Therefore

$$P[\text{error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

For antipodal signalling, the optimum decisions are performed by comparing the samples of the matched filter’s output (sampled at exactly integer multiples of the bit duration) with a threshold 0. Of course such an optimum decision rule does not change if the impulse response of the matched filter is scaled by a positive constant.
Scaling the matched filter’s impulse response \( h_R(t) \) does not change the receiver performance because it scales both signal and noise components by the same factor, leaving the signal-to-noise ratio (SNR) of the decision variable unchanged!

In the above block diagram, \( h_R(t) = \kappa p(T_b - t) \). We have been using \( \kappa = 1/\sqrt{E} \) in order to represent the signals on the signal space diagram (which would be at \( \pm \sqrt{E} \)) and to conclude that the variance of the noise component is exactly \( N_0/2 \).

For an arbitrary scaling factor \( \kappa \), the signal component becomes \( \pm \kappa E \), while the variance of the noise component is \( N_0/2 \kappa^2 E \). Thus, the SNR is

\[
\text{SNR} = \frac{\text{Signal power}}{\text{Noise power}} = \frac{(\pm \kappa E)^2}{\frac{N_0}{2} \kappa^2 E} = \frac{2E}{N_0}, \quad \text{(independent of } \kappa!\text{)}
\]

In terms of the SNR, the error performance of antipodal signalling is

\[
P[\text{error}] = Q\left( \sqrt{\frac{2E}{N_0}} \right) = Q\left( \sqrt{\text{SNR}} \right)
\]

In fact, it can be proved that the receive filter that maximizes the SNR of the decision variable must be the *matched* filter. It is important to emphasize that the *matching* property here concerns the shapes of the impulse responses of the transmit and receive filters.
Chapter 5: Optimum Receiver for Binary Data Transmission

Outputs of the Matched/Mismatched Filters (No-Noise Scenario)

Clean received signals for rect and half-sine (HS) shaping filters

Output of a matched filter: rect/rect matching

Outputs of HS/HS matched filter (red) and HS/rect mismatched filter (pink)

When the matched filter is used, sampling at exact multiples of the bit duration maximizes the power of the signal component in the decision variable, hence maximizing the SNR. A timing error (imperfect sampling) would reduce the power of the signal component, hence reducing the SNR, hence degrading the performance, i.e., increasing $P[\text{error}]$.

When the receive filter is not matched to the transmit filter, the power of the signal component and the SNR are not maximized, even under perfect sampling!
Chapter 5: Optimum Receiver for Binary Data Transmission

Antipodal Baseband Signalling with Rectangular Pulse Shaping

![Diagram](image)

- **Bits**: Two voltage levels, 0 and 1.
- **LUT**: Lookup Table with levels -V and +V.
- **Pulse Shaping Filter**: Transforms the input signal.
- **Transmitter**: Produces the final output signal.

- **Receiver**: Receives the signal and uses a matched filter to obtain the decision variable.
- **Decision**: Determines the transmitted bit based on the received signal.

Mathematical expressions:
- \( r(t) \): Received signal
- \( h(t) = p(T_b - t) \): Matched filter impulse response
- \( x(t) \): Output of the matched filter
- \( x(kT_b) \): Decision variable
- \( \hat{b}_k \): Estimated bit

---

EE456 – Digital Communications
Antipodal Baseband Signalling with Half-Sine Pulse Shaping

**Bits**

0 \rightarrow 0 \rightarrow b(t) \rightarrow \text{Pulse shaping filter} \rightarrow s(t) \rightarrow \pm V

- **Voltage levels**
  - 0 \rightarrow 0 \rightarrow b(t)
  - 1 \rightarrow 0 \rightarrow b(t)

**Matched filter**

\( h(t) = p(t) \)

\( r(t) \rightarrow x(t) \rightarrow x(kT_b) \rightarrow \hat{b}_k \)

\( h(t) = p(T_b - t) \)

**Receiver**
Applicable to *any* binary modulation with *arbitrary* a priori probabilities, but restricted to *statistically independent* bits.

\[
s_T(t) = \sum_{k=-\infty}^{\infty} g_k(t), \quad g_k(t) = \begin{cases} s_1(t - kT_b), & \text{with probability } P_1 \\ s_2(t - kT_b), & \text{with probability } P_2 \end{cases}
\]

The derivation on the next slide shows that:

\[
S_{s_T}(f) = \frac{P_1P_2}{T_b} |S_1(f) - S_2(f)|^2 + \sum_{n=\infty}^{\infty} \left| \frac{P_1S_1 \left( \frac{n}{T_b} \right) + P_2S_2 \left( \frac{n}{T_b} \right)}{T_b} \right|^2 \delta \left( f - \frac{n}{T_b} \right).
\]
\[ s_T(t) = E\{s_T(t)\} + s_T(t) - E\{s_T(t)\} = v(t) + q(t) \]

\[ v(t) = E\{s_T(t)\} = \sum_{k=-\infty}^{\infty} [P_1 s_1(t - kT_b) + P_2 s_2(t - kT_b)] \]

\[ S_v(f) = \sum_{n=-\infty}^{\infty} |D_n|^2 \delta \left( f - \frac{n}{T_b} \right), \quad D_n = \frac{1}{T_b} \left[ P_1 S_1 \left( \frac{n}{T_b} \right) + P_2 S_2 \left( \frac{n}{T_b} \right) \right], \]

\[ S_v(f) = \sum_{n=-\infty}^{\infty} \left| \frac{P_1 S_1 \left( \frac{n}{T_b} \right) + P_2 S_2 \left( \frac{n}{T_b} \right)}{T_b} \right|^2 \delta \left( f - \frac{n}{T_b} \right). \]

To calculate \( S_q(f) \), apply the basic definition of PSD:

\[ S_q(f) = \lim_{T \to \infty} \frac{E\{|G_T(f)|^2\}}{T} = \cdots = \frac{P_1 P_2}{T_b} |S_1(f) - S_2(f)|^2. \]

\[ S_{s_T}(f) = \frac{P_1 P_2}{T_b} |S_1(f) - S_2(f)|^2 + \sum_{n=-\infty}^{\infty} \left| \frac{P_1 S_1 \left( \frac{n}{T_b} \right) + P_2 S_2 \left( \frac{n}{T_b} \right)}{T_b} \right|^2 \delta \left( f - \frac{n}{T_b} \right). \]

For the special, but important case of antipodal signalling, \( s_2(t) = -s_1(t) = p(t) \), and equally likely bits, \( P_1 = P_2 = 0.5 \), the PSD of the transmitted signal is solely determined by the Fourier transform of \( p(t) \):

\[ S_{s_T}(f) = \frac{|P(f)|^2}{T_b} \]
Chapter 5: Optimum Receiver for Binary Data Transmission

Baseband Message Signals with Different Pulse Shaping Filters

Information bits or amplitude levels

Output of the transmit pulse shaping filter – Rectangular

Half-sine pulse shaping filter

SRRC pulse shaping filter ($\beta = 0.5$)

PSD - Rectangular

PSD - Half-sine

PSD - SRRC (30 symbols long)
Chapter 5: Optimum Receiver for Binary Data Transmission

Building A Binary (Antipodal) Comm. System in Labs #4 and #5

Figure 1: Block diagram of the transmitter.
Figure 2: Block diagram of the receiver.